What is the fate of a black hole embedded in an expanding universe?

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Abstract

Within a large class of exact solutions of the Einstein equations describing a black hole embedded in a Friedmann universe it is shown that, under certain assumptions, only those with comoving Hawking-Hayward quasi-local mass are generic, in the sense that they are late-time attractors.

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1 Introduction

There is currently much interest in dynamical horizons [1] and solutions of the Einstein equations modelling dynamical black holes [2]. An example of such situations is that of a black hole embedded in an expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. While the effect of the cosmic expansion on the local dynamics of astrophysical black holes is completely negligible today, this may not have been the case for primordial black holes in the very early universe [3]. Moreover, the Hawking temperature and thermodynamics of time-dependent horizons appear to be interesting subjects for semiclassical gravity [2, 4, 5, 6].

With these motivations in mind, we have found exact solutions describing black holes embedded in FLRW spaces [7, 8]. For simplicity, and to reproduce the observed universe, we will limit ourselves to consider spatially flat expanding FLRW universes as backgrounds. These can be used as realistic solutions for certain situations and as toy models for others. Here we are concerned with two classes of solutions found in [7] and discussed in [8]. These solutions generalize the McVittie metric and can be written, in isotropic coordinates (t, r, θ, φ) as

$$\begin{aligned} ds^2 &= -\frac{\left[1-\frac{M(t)}{2a(t)r}\right]^2}{\left[1+\frac{M(t)}{2a(t)r}\right]^2}\, dt^2 + a^2\left(t\right)\left[1+\frac{M(t)}{2a\left(t\right)r}\right]^4 \\ &\quad \cdot \left(dr^2 + r^2 d\Omega^2\right) \,, \end{aligned} \tag{1.1}$$

where $d\Omega^2$ denotes the line element on the unit 2-sphere. The McVittie metric [9] corresponds to $M = M_0$ =const., while here $M(t)$ is an *a priori* arbitrary function of the cosmic time t which is positive and continuous with its first derivative. The constancy of M in the McVittie metric expresses the McVittie assumption that $G_0^1 = 0$, *i.e.*, that the component of the stress-energy tensor $T_0^1 = 0$, hence there is no radial energy flow onto or from the central object (no radial accretion or excretion) [9]. As shown in [7], the metric coefficient $M(t)$ is the Hawking-Hayward quasi-local mass [10], which should be regarded as the physical mass of the central black hole. In conjunction with the fact that the size of the McVittie central object does not change during the expansion of the universe,[1] the McVittie no-accretion condition $M(t)$ =const. simply enforces the constancy of this mass.

[1] The mass of the central object can, in principle, change because of two processes: the expansion of the object, which then swallows cosmic fluid, and a radial flow onto the object from far away or from the object.

Over the years, it became clear that, with the exception of the Schwarzschild-de Sitter metric, the McVittie spacetime can not describe a central black hole (or a regular strong field object) because of a spacelike singularity at $r = M_0/2$, corresponding to a diverging pressure of the cosmic perfect fluid sourcing this metric [11, 12, 13].

Exact solutions discovered later [14, 15, 16, 17], such as the Sultana-Dyer solution [16], if free from singularities (other than the central black hole singularity and the usual cosmological ones), suffer from negative energy densities and, in addition, are limited in the type of FLRW background in which they can be embedded (*e.g.*, only an $a \propto t^{2/3}$ scale factor for the Sultana-Dyer solution), and in the type of matter that can source them (*e.g.*, a mixture of two perfect fluids, one of which is a null dust, for the Sultana-Dyer metric).

The solutions (1.1) presented in [7] have the advantages of being free of singularities (apart from the central black hole singularity and the usual Big Bang or Big Rip cosmological singularities), and that the fluid source is relatively simple: a single imperfect fluid with a radial heat flux, described by the stress-energy tensor

$$T_{\mu\nu} = (P+\rho)\, u_\mu u_\nu + P g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \;, \tag{1.2}$$

where $u^\mu = \left(|g_{00}|^{-1/2}, 0, 0, 0, \right)$ is the fluid four-velocity and $q^\mu = (0, q, 0, 0,)$ is the radial heat current.

In [7], emphasis was put on a class of solutions with $M(t) = M_0 a(t)$, *i.e.*, with comoving quasi-local mass. These solutions possess a conformal Killing horizon and, in this sense, resemble the Sultana-Dyer solution which is constructed by conformally transforming the Schwarzschild metric with the scale factor of a dust-dominated universe as conformal factor, but requires a two-fluid source [16]. The conformally expanding solutions of [7] were also used in [8] and [6]. In [8], they were studied with emphasis on the behaviour of the black hole apparent horizon in universes dominated by phantom dark energy with equation of state $P < -\rho$. A second class of solutions with arbitrary function $M(t)$ was also discussed in [8]. Here we show that the solutions of this second, and apparently more general, class can be attracted at late times toward the "comoving mass" solutions during the expansion of the universe. Therefore, future research can safely focus on this much simpler class of comoving solutions.

2 Comoving quasi-local mass solutions as late-time attractors

Following the notations of [8], we begin by switching to the areal radius $\tilde{r} \equiv r\left(1+\frac{M(t)}{2a(t)r}\right)^2$ and then to its comoving version $R \equiv a\tilde{r}$, in terms of which the metric (1.1) is turned into the Painlevé-Gullstrand form

$$
\begin{aligned}
ds^2 &= -\left[1-\frac{2M}{R}-\frac{\left(HR+\dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right)^2}{1-\frac{2M}{R}}\right]dt^2 \\
&+ \frac{dR^2}{1-\frac{2M}{R}}+R^2d\Omega^2-\frac{2}{1-\frac{2M}{R}}\left(HR+\dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right)dt\,dR\,,
\end{aligned}
\tag{2.1}
$$

where $m(t) \equiv M(t)/a(t)$. Defining $A(t,R) \equiv 1-2M/R$ and $C(t,R) \equiv HR+\dot{m}a\sqrt{\frac{\tilde{r}}{r}}$ and introducing a new time coordinate T defined by

$$
dT = \frac{1}{F}\left(dt+\frac{C}{A^2-C^2}\,dR\right)\,, \tag{2.2}
$$

where $F\left(T(t,R),R\right)$ is an integrating factor that makes dT an exact differential and satisfies

$$
\partial_R\left(\frac{1}{F}\right) = \partial_t\left[\frac{C}{F(C^2-A^2)}\right]\,, \tag{2.3}
$$

one cancels the cross terms in $dR\,dT$ and casts the line element in the Nolan gauge

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2M}{R}\right)\left[1-\frac{\left(HR+\dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right)^2}{\left(1-\frac{2M}{R}\right)^2}\right]F^2dT^2 \\
&+\left(1-\frac{2M}{R}\right)^{-1}\left[1-\frac{\left(HR+\dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right)^2}{\left(1-\frac{2M}{R}\right)^2}\right]^{-1}dR^2+R^2d\Omega^2\,.
\end{aligned}
\tag{2.4}
$$

The black hole apparent horizon is located at the smallest root of the equation

$$
HR+ma\left(1+\frac{m}{2r}\right)\left(\frac{\dot{M}}{M}-H\right) = 1-\frac{2M}{R}\,. \tag{2.5}
$$

Since $r = r(R)$, this is an implicit equation for the horizon radius. The expression $\left(\frac{\dot{M}}{M} - H\right) = \frac{\dot{m}}{m}$ describes the deviation of the rate of change of the quasi-local black hole mass from the Hubble rate, *i.e.*, it measures the deviation from stationary accretion *with respect to the background*. This expression vanishes for comoving mass solutions of the first class, which have $M(t) = M_0 a(t)$.

We are now going to show that comoving mass solutions are generic under certain assumptions and all other solutions of the type (1.1) approach them at late times. We assume 1) that the universe expands, 2) that $m(t) \geq 0$, and 3) that this function is continuous with its first derivative. Let us use the radial variable $\tilde{r} \equiv R/a$. Then, eq. (2.5) becomes

$$H\tilde{r} + \frac{2m}{\tilde{r}a} = -\dot{m}\left(1 + \frac{m}{2r}\right) + \frac{1}{a} \, . \tag{2.6}$$

Since $m \geq 0$, the left hand side is clearly non-negative at all times, so $\dot{m}\left(1 + \frac{m}{2r}\right) < \frac{1}{a}$. Therefore, in an expanding universe in which $a \to +\infty$, and given that $1 + \frac{m}{2r} > 0$, it is $\dot{m}_\infty \equiv \lim_{t\to+\infty} \dot{m}(t) \leq 0$. If $\dot{m}_\infty = 0$, the black hole becomes asymptotically comoving. Let us focus on the case $\dot{m}_\infty < 0$. Then, there exists a time $\bar{t}$ such that, for all times $t > \bar{t}$, it is $\dot{m}(t) < 0$. There are only two possibilities in this case: since $m(t) \geq 0$, either
a) $m(t)$ reaches the value zero at a finite time t_* with derivative $\dot{m}_* \equiv \dot{m}(t_*) < 0$, or
b) $m(t) \to m_0$ =const. with $\dot{m}(t) \to 0$, *i.e.*, $m(t)$ has a horizontal asymptote.

In case a) one has, at $t = t_*$, $HR = |\dot{m}_*|\, a + 1$, which yields the radius of the black hole apparent horizon at t_*

$$r_* \equiv r_{horizon}(t_*) = \frac{1}{H(t_*)}\left(|\dot{m}_*| + \frac{1}{a}\right) \, . \tag{2.7}$$

Late in the history of the universe, we have a black hole of zero quasi-local mass $M(t_*) = a(t_*)m(t_*)$ but finite radius r_*. As time evolution continues, one would have negative mass M and finite radius of the black hole apparent horizon. This does not make sense physically and, therefore, the case $m(t_*) = 0$ with $m(t > t_*) < 0$ is ruled out.

We are left with case b) in which $\dot{m}(t) \to 0$ at late times (*i.e.*, $t \to +\infty$ if the cosmic expansion continues forever, or $t \to t_{rip}$ if a Big Rip occurs at the time t_{rip}). The physical meaning is that, at late times, the rate of increase of the black hole mass is at most the Hubble rate and the black hole becomes comoving. This conclusion is, of course, not valid at early times, at which the term $1/a$ in eq. (2.6) does not tend to zero.

As a special case of b), it is possible that m_0 is zero, in which case the solution reduces to a FLRW universe without inhomogeneities and can be interpreted as a black

hole that evaporates completely[2]. This possibility is non-trivial from the physical point of view.

Physically relevant situations in which the black hole does not become comoving, which are not included in the previous description, may arise if the assumptions are relaxed. For example, if the assumption of continuity of $\dot{m}(t)$ is dropped, one can contemplate the situation in which $m(t) \to 0$ in a finite time t_{ev} and

$$\dot{m}(t) \begin{cases} < 0 & \text{if } t \leq t_{ev} \,, \\ = 0 & \text{if } t > t_{ev} \,, \end{cases} \tag{2.8}$$

Such a spacetime would have a continuous metric, discontinuous Christoffel symbols, and distributional curvature (in analogy to exact *pp*-waves) and could represent a black hole evaporating as $m \to 0$ when $t \to t_{ev}$. A detailed study of this possibility will be pursued elsewhere.

3 Discussion and conclusions

The two classes of exact solutions of the Einstein equations recently proposed in [7, 8] and describing a black hole embedded in a spatially flat FLRW universe and accreting cosmic fluid, are of interest to study dynamical horizons and their thermodynamics. Such solutions are useful as testbeds for various conjectures on time-dependent horizons, and are relatively rare. It is therefore important to look for simple solutions which do not suffer from problems such as unphysical singularities, negative energy densities, or being sourced by exotic forms of matter that could hide the physics under investigation.

Under the assumptions 1) that the universe expands; 2) that the mass parameter m is non-negative; and 3) that the function $m(t)$ is continuous with its first derivative, we have shown that only the first class of solutions considered in [7, 8], in which the Hawking-Hayward quasi-local mass is comoving, $M(t) = m_0 a(t)$, is generic, in the sense that these solutions act as late-time attractors for all the solutions of the second class (exceptions are black holes with mass asymptotically going to zero, which cannot be called "comoving"). Therefore, future research can focus on the first class of solutions, which are simpler (that they are much simpler than solutions of the second class was demonstrated in the study of their black hole and cosmic apparent horizons in [8]).

Apart from the interest in time-dependent horizons and their thermodynamics, a lesson to be learned is that, in an expanding universe, self-gravitating objects eventually

[2]An obstacle to this interpretation is that the radial flow considered in these solutions is not described by a null vector. A generalization will be pursued elsewhere.

tend to participate in the global expansion and to align their evolutionary dynamics with that of the cosmic substratum. The situation studied here for black holes is very similar to that investigated for wormholes in Ref. [18]. There, using exact solutions describing a wormhole embedded in a FLRW universe, it was found that even if a wormhole starts expanding much faster (or much slower) than the cosmic substratum, eventually it catches up with the cosmic expansion and becomes comoving.

At present, it is not clear whether the metric (2.5) is the most general spherically symmetric solution describing a black hole embedded in a spatially flat FLRW background, in the same sense that the Schwarzschild solution is the most general vacuum, spherically symmetric and asymptotically flat black hole metric. This can only be decided by a separate analysis and verified by perturbation studies, which will be pursued elsewhere.

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